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On Multifractality and Fractional Derivatives

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It is shown phenomenologically that the fractional derivative $\xi = D^\alpha u$ of order α of a multifractal function has a power-law tail $\propto |\xi|^{-p_\star}$ in its cumulative probability, for a suitable range of α's. The exponent is determined by the condition $\zeta_{p_\star} = \alpha p_\star$, where ζ_p is the exponent of the structure function of order p. A detailed study is made for the case of random multiplicative processes (Benzi *et al.*, *Physica D* **65**:352 (1993)) which are amenable to both theory and numerical simulations. Large deviations theory provides a concrete criterion, which involves the departure from straightness of the ζ_p graph, for the presence of power-law tails when there is only a limited range over which the data possess scaling properties (e.g., because of the presence of a viscous cutoff). The method is also applied to wind tunnel data and financial data.

KEY WORDS: Turbulence; multifractals; large deviations; finance.

1. INTRODUCTION

Multifractality for functions was introduced by Parisi and Frisch[(1)] to interpret experimental results of Anselmet *et al.* on fully developed turbulence.[(2)] We show in this paper that multifractality is connected to the tail behavior of fractional derivatives (to be defined precisely later). First, in Section 2 we give a simple phenomenological argument predicting power-law tails in probability distributions of fractional derivatives and suggesting practical constraints on their observability. In Section 3 we consider the case of a class of synthetic multifractal functions, the random multiplicative processes of Benzi *et al.*,[(3)] for which theoretical results are obtained by

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relating them to the theory of random linear maps.[4] In particular, using large deviations theory, we can predict the number of generations in a given random multiplicative process needed to observe power-laws tails in probabilities of fractional derivatives. Numerical experiments with such processes are presented in Section 3.3. In Section 4 we analyze high-Reynolds number wind tunnel turbulence data and financial data; we show that the former, having only weak deviations from self-similar behavior à la Kolmogorov 1941, are unlikely to display power-law tails at accessible Reynolds numbers. Section 5 gives the conclusions. In this paper we consider exclusively multifractal functions; the case of multifractal measures, arising, e.g., from chaotic dynamical systems or the dissipation of turbulent flows (see, e.g., ref. 5), will be addressed elsewhere.

2. PHENOMENOLOGY

A (homogeneous random) function $u(x)$ with $x \in \mathbb{R}^d$ is called multifractal if (for some real interval $0 < h_{\min} \leqslant h \leqslant h_{\max} < 1$) there is a continuum of sets $\mathcal{S}_h$ of fractal dimension $D(h)$ such that, when x belongs to $\mathcal{S}_h$, the function u has a Hölder-like behavior with exponent $0 < h < 1$.[1] In terms of increments this is expressed as

$$|\delta u(x, r)| \propto |r|^h, \qquad r \to 0, \quad x \in \mathcal{S}_h, \tag{1}$$

where $\delta u(x, r) = u(x+r) - u(x)$. If we take $D(h)$ to be the covering dimension (see ref. 6, Section 8.5.1 and references therein) and use instead the codimension $F(h) \equiv d - D(h)$, where d is the dimension of space, and then "thicken" the set $\mathcal{S}_h$ by covering it with balls of small radius ρ, into the set $\mathcal{S}_h^\rho$, we have

$$\mathrm{Prob}\{x \in \mathcal{S}_h^\rho\} \propto \rho^{F(h)}, \qquad \rho \to 0. \tag{2}$$

For the case of an ergodic random function, this probability can also be interpreted as a fraction of space. Note that by using codimensions the extension from the one-dimensional to the multi-dimensional case is straightforward (just replace scalars used throughout this paper by vectors). For more background on multifractals see ref. 6 and references therein.

We turn now to fractional derivatives. The definition used here for the fractional derivative $D^\alpha u(x)$ of order $0 \leqslant \alpha < 1$ of a function $u(x)$ is to multiply its Fourier transform $\hat{u}(k)$ by $|k|^\alpha$. This might be called, more correctly, the fractional (negative) Laplacian of order $\alpha/2$. In physical space our fractional derivative may be written, in the one-dimensional case, as

$$D^{\alpha}u(x) \equiv -\frac{\alpha}{\pi}\sin\frac{\alpha\pi}{2}\,\Gamma(\alpha)\text{ P.V.}\int\frac{u(x+r)-u(x)}{|r|^{\alpha+1}}\,dr, \tag{3}$$

where $\Gamma(\alpha)$ is the Gamma function and P.V. denotes a Cauchy principal value. The Fourier-space definition is very convenient for periodic functions. When working with non-periodic experimental data, windowing may be necessary before the Fourier transformation is applied. Note that other definitions of the fractional derivative, for example that based on the ordinary derivative of the the Riemann–Liouville fractional integral,[(7)] do not produce substantially different results, as far as our work is concerned.

It is clear from (3) that, at a point x_0 at which $u(x)$ has Hölder exponent $h > \alpha$ the fractional derivative $D^{\alpha}u(x_0)$ is finite (provided the function u is, say, bounded). If $\alpha \geqslant h$ the fractional derivative will generally be infinite. This is also the case with the definition of fractional derivatives used in ref. 8. But what takes place in the *neighborhood* of such a point? Let us first consider the case of an isolated non-oscillatory singularity of exponent h at x. That is, we have $|u(x+r)-u(x)| \propto |r|^{h}$ for small r. By substitution into (3) it is easily checked that, with a suitable constant $B > 0$

$$|D^{\alpha}u(y)| \sim B\,|y-x|^{h-\alpha}, \qquad y \to x, \tag{4}$$

a relation which could have been guessed by simple dimensional analysis. Since $h-\alpha < 0$, the fractional derivative becomes very large near x.

What happens if instead of isolated singularities we have multifractal non-oscillatory singularities? By a standard argument of multifractal analysis, in the phenomenological presentation of ref. 1 (see also ref. 6, Section 8.5), we still use (4) for determining the contribution of points $x \in \mathcal{S}_h$ to $D^{\alpha}u(y)$ calculated at points which are near $\mathcal{S}_h$ without belonging to this set. It is now very simple to estimate the probability to have $|D^{\alpha}u| > \xi$ for large positive ξ. From (4), the contribution from a given $h < \alpha$ to $|D^{\alpha}u(y)|$ will be greater than ξ provided that

$$|y-x| < \left(\frac{\xi}{B}\right)^{-\frac{1}{\alpha-h}}. \tag{5}$$

By (2), this has probability

$$\text{Prob}\{|D^{\alpha}u| > \xi\} \propto |y-x|^{F(h)} \propto \xi^{-\frac{F(h)}{\alpha-h}}. \tag{6}$$

This is a power law in ξ. As we vary the Hölder exponent h, the exponent of this power law changes. When we sum the contributions of the various h's (with suitable regular weights $d\mu(h)$ which we need not specify) we get an integral which, when evaluated by Laplace's method, gives us leading-order power-law behavior for the tail probability:

$$\mathrm{Prob}\{|D^{\alpha}u| > \xi\} \propto \xi^{-p_\star}, \tag{7}$$

$$p_\star(\alpha) = \inf_{h<\alpha} \frac{F(h)}{\alpha - h}. \tag{8}$$

We shall now show that $p_\star$ can be easily found from the scaling exponents ζ_p of structure functions. We remind the reader that, when $u(x)$ is a homogeneous multifractal function, the moments of its increments, called structure functions, are defined as

$$S_p(r) \equiv \langle (|\delta u(r)|)^p \rangle, \qquad p \geqslant 0. \tag{9}$$

They follow power laws at small r's:

$$S_p(r) \propto |r|^{\zeta_p}, \tag{10}$$

where the up convex function ζ_p and the down convex function $F(h)$ are Legendre–Fenchel transforms of each other[1, 6]

$$F(h) = \sup_p \,(\zeta_p - ph), \qquad \zeta_p = \inf_h \,(ph + F(h)). \tag{11}$$

Using (11) in (8) and interchanging the orders of the infimum and the supremum, we obtain

$$p_\star = \sup_{p \geqslant 0} \inf_{h<\alpha} \frac{\zeta_p - ph}{\alpha - h}. \tag{12}$$

The infimum over $h<\alpha$ is $-\infty$ if $\zeta_p - p\alpha < 0$, it equals p if $\zeta_p - p\alpha = 0$ and is less than this value if $\zeta_p - p\alpha > 0$. Thus $p_\star(\alpha)$ is the supremum of all p's such that $\zeta_p - p\alpha \geqslant 0$; in other words, it is the solution of

$$\zeta_{p_\star} = p_\star(\alpha)\,\alpha, \tag{13}$$

which is unique in view of the convexity of ζ_p and vanishing for $p=0$ (a consequence of (9)). Thus, the exponent $p_\star(\alpha)$ of the tail probability of the fractional derivative of order α can be obtained from the graph of ζ_p by the simple construction shown in Fig. 1: a line through the origin of slope α intersects the graph at $p = p_\star(\alpha)$.

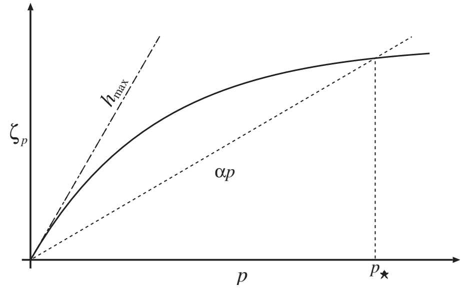

Fig. 1. Geometrical determination of the exponent $p_\star(\alpha)$ of the power-law tail from the graph of ζ_p, the structure-function exponents. h_{max} is the maximum Hölder exponent; h_{min}, the minimum Hölder exponent, is the (algebraically) smallest slope, not shown.

This construction has a number of interesting consequences which are now presented. First, recall that

$$h_\star = \left.\frac{d\zeta_p}{dp}\right|_{p=p_\star}. \tag{14}$$

(This follows from the observation that $F(h)$, if down convex, can be recovered from ζ_p by another Legendre transformation.) If there is a minimum Hölder exponent h_{min} and maximum exponent h_{max}, it is clear that α has to be between those two values for $p_\star$ to exist. If $\alpha \leqslant h_{min}$ we do not expect any power-law tail at all (at least not by the mechanism considered here). If $\alpha > h_{max}$ we expect $D^\alpha u(x)$ to be infinite almost everywhere. $D^\alpha u(x)$ can then be made finite if we assume that there is some ultraviolet cutoff, e.g., a viscous cutoff. If so, the tail probability of $D^\alpha u(x)$ can be estimated by the same kind of phenomenological arguments used for the tail probability of velocity gradients in turbulence.[6, 9, 10]

Next, we can use the construction to estimate how far the power-law tail (if present at all) is expected to extend when multiscaling holds only over a finite range of scales. By (4) the fractional derivative stemming from the Hölder exponent $h_\star$ is given by

$$|D^\alpha u(y)| \sim B\,|y-x|^{-C(\alpha)}, \tag{15}$$

$$C(\alpha) = \alpha - h_\star = \frac{\zeta_{p_\star}}{p_\star} - \left.\frac{d\zeta_p}{dp}\right|_{p=p_\star}. \tag{16}$$

$C(\alpha)$ will be called the *multifractality parameter*. In the next sections we shall see that the presence of a power-law tail requires in practice

$$nC(\alpha) \geqslant 10, \tag{17}$$

where n is the number of octaves in the scaling range (e.g., in the inertial range for turbulence). If the range of spatial scales is somewhat limited, it is crucial that $C(\alpha)$ be as large as possible. In those instances where the graph of ζ_p is close to a straight line through the origin, the two terms on the r.h.s. of (16) will nearly cancel and $C(\alpha)$ will be very small. Clearly, having a graph departing strongly from a straight line will help in seeing power-law tails.

All the material presented in this phenomenological section is, by definition, "soft." Is it possible to give harder evidence? One way is to work with Burgers turbulence in the limit of vanishing viscosity (ref. 11 and references therein). The singularities (mostly shocks) are then isolated and it is easy to make our phenomenological arguments rigorous. Solutions to the Burgers equation are however bifractal, not truly multifractal. In the next section we discuss a simple example of a truly multifractal function.

3. THE CASE OF THE MULTIFRACTAL RANDOM MULTIPLICATIVE PROCESS

In 1993 Benzi *et al.*[3] introduced an explicit method for constructing multifractal random functions with arbitrary ζ_p. Such functions are constructed by iterating suitable random maps and will be called here "random multiplicative processes" (rmp). Our method of construction differs somewhat from that of ref. 3 but is basically equivalent.

We first define the local rmp as a real random function on the real line. Let $\phi(x)$ be an indefinitely differentiable function with rapid decrease at infinity and vanishing space integral. In practice we shall take the Mexican hat function of width σ used in ref. 3

$$\phi(x) \equiv -\frac{d^2}{dx^2} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{18}$$

Let m be a random variable, called the *random multiplier*, with symmetric distribution and finite moments of all orders, characterized by its cumulative distribution

$$P(\eta) \equiv \text{Prob}\{m > \eta\} \tag{19}$$

and its probability density

$$p(\eta) \equiv -\frac{dP(\eta)}{d\eta} = p(-\eta). \tag{20}$$

We shall also need independent identically distributed copies of m, denoted $m_n^{(1)}$ and $m_n^{(2)}$ ($n=0, 1,...$). (Henceforth the words "identically distributed" are understood.) We define now a sequence of random functions $w_n(x)$ recursively by

$$w_0(x)=\phi(x), \tag{21}$$

$$w_{n+1}(x)=\phi(x)+m_n^{(1)}w_n^{(1)}(2x)+m_n^{(2)}w_n^{(2)}(2x-1), \tag{22}$$

$$n=0, 1,...,$$

where $w_n^{(1)}(x)$ and $w_n^{(2)}(x)$ are independent copies of $w_n(x)$. The almost sure pointwise limit for $n\to\infty$ of $w_n(x)$, if it exists, is denoted by $w(x)$ and satisfies obviously

$$w(x)\overset{\text{law}}{=}\phi(x)+m^{(1)}w^{(1)}(2x)+m^{(2)}w^{(2)}(2x-1), \tag{23}$$

where $\overset{\text{law}}{=}$ designates equality in law. It may be checked that our random map definition is equivalent to that given in ref. 3, which involves an infinite series.

Assuming the limit to exist, we now define the global rmp as

$$u(x)\equiv\sum_{i=-\infty}^{i=+\infty} w^{(i)}(x+x_i), \tag{24}$$

where the $w^{(i)}(x)$ are independent copies of $w(x)$ and the x_i are Poisson-distributed points on the real line with density c.

The construction of the global rmp from the local one, which differs slightly from that of ref. 3, guarantees statistical homogeneity. Denoting statistical means (expectation values) by angular brackets, we can relate the characteristic functionals of $u(x)$ and $w(x)$. Indeed, we have

$$\left\langle \exp\left\{\int_{\mathbb{R}} \imath\varphi(x)\,u(x)\,dx\right\}\right\rangle$$

$$=\exp\left\{c\int_{\mathbb{R}}\left[\left\langle\exp\left(\int_{\mathbb{R}}\imath\varphi(x)\,w(x+y)\,dx\right)\right\rangle-1\right]dy\right\}, \tag{25}$$

where $\varphi(x)$ is a test function and $\int_{\mathbb{R}}$ denotes integration from $-\infty$ to $+\infty$. (To establish (25) it is convenient to assume that there are N points x_i distributed uniformly in the interval $[-L, +L]$ and then let $N\to\infty$, $L\to\infty$

and $N/(2L) \to c$.). A special case of (25) relates just the characteristic functions (which for u does not depend on x):

$$\langle e^{\imath z u} \rangle = \exp\left\{ c \int_{\mathbb{R}} (\langle e^{\imath z w(y)} \rangle - 1)\, dy \right\}. \tag{26}$$

The fractional derivative $D^{\alpha} u(x)$ is obtained by replacing, in (24), $w(x)$ by $D^{\alpha} w(x)$. Applying D^{α} to (23), we find that $D^{\alpha} w(x)$ satisfies

$$w_{\alpha}(x) \stackrel{\text{law}}{=} \phi_{\alpha}(x) + 2^{\alpha} m^{(1)} w_{\alpha}^{(1)}(2x) + 2^{\alpha} m^{(2)} w_{\alpha}^{(2)}(2x-1), \tag{27}$$

$$w_{\alpha}(x) \equiv D^{\alpha} w(x), \qquad \phi_{\alpha}(x) \equiv D^{\alpha} \phi(x), \tag{28}$$

which is basically the same as (23) with ϕ changed into $D^{\alpha}\phi$ and the random multiplier rescaled by a factor 2^{α}.

Various theoretical results for structure functions were obtained for rmp's in ref. 3. In particular it was shown that the structure function of order $p \geqslant 0$ has (with our notation) the scaling exponent

$$\zeta_p = -\ln_2 \langle |m|^p \rangle. \tag{29}$$

Here, we need new results for probability distributions. It turns out that there is a class of space-independent linear random maps (Kesten maps) whose probability distributions are very closely connected to those of the rmp's. They are discussed in the next subsection.

3.1. Kesten's Random Maps

Consider the sequence of random variables defined recursively by

$$w_0 = 1, \tag{30}$$

$$w_{n+1} = 1 + m_n w_n, \tag{31}$$

$$n = 0, 1, \ldots,$$

where the m_n's are independent copies of the random multiplier m, with the same definition as used above. The limit of w_n for $n \to \infty$, if it exists, has the same distribution as

$$w = 1 + m_1 + m_1 m_2 + \cdots + m_1 m_2 \cdots m_n + \cdots \tag{32}$$

Kesten has obtained a number of important results for a more general class of maps where the multipliers may be random matrices.[4] In its scalar form

(31) has attracted the attention of statistical physicists interested in disordered systems or finance.[12–14] Results on the existence of invariant measures for Kesten maps and more general nonlinear random maps are found for example in refs. 15–17. Here, we are only interested in the scalar linear map defined by (31), which we shall call the "Kesten map" associated to the rmp.

We now summarize a few important results for Kesten maps without giving proofs, but we shall give occasional hints. Let

$$\lambda \equiv \langle \ln |m| \rangle \tag{33}$$

be the Lyapunov exponent of the map. If $\lambda > 0$ the w_n's run away to infinity almost surely. ($\ln |m_1 m_2 \cdots m_n|$ grows approximately as λn.) If $|m| \leqslant a < 1$ almost surely, then the series (32) converges to a value $< 1/(1-a)$. Hence, the tail behavior of w at large values is trivial. When $\lambda < 0$ and there is a non-trivial tail, the nature of the invariant measure (the distribution of w) depends in particular on the average slope $\langle |m| \rangle$ of the map. If $\langle |m| \rangle > 1$ it may be shown that the invariant measure is absolutely continuous, i.e., that there is a finite probability density for w. (This may still hold, but only in special instances, for $\langle |m| \rangle < 1$.) The most important result is that, for $\langle |m| \rangle > 1$ and $\lambda < 0$, there is a power-law tail:

$$\text{Prob}\{|w| > \xi\} \propto \xi^{-p_\star}, \qquad \xi \to \infty, \tag{34}$$

where $p_\star$ is the single positive number such that

$$\langle |m|^{p_\star} \rangle = 1. \tag{35}$$

(Take the absolute value of (31), raise it to the pth power, average and neglect the contribution of the additive 1 when p is close to $p_\star$ and just below, so that this moment comes predominantly from the tail behavior of w.)

Now we give an alternative derivation of this power-law behavior, based on large deviations, which is in spirit very close to the phenomenological derivation of power laws in Section 2. For background on large deviations see, e.g., refs. 18 and 19 (an elementary introduction may be found in Section 8.6.4 of ref. 6).

Consider the nth term on the r.h.s. of (32). Since the m variables have symmetric distributions, it may written

$$\pm |m_1|\,|m_2| \cdots |m_n| = \pm 2^{-(a_1 + a_2 + \cdots + a_n)}, \qquad a_i \equiv -\ln_2 |m_i|, \tag{36}$$

where the plus and minus signs are taken with equal probabilities $1/2$.

First, let us assume that just one term dominates the tail behavior of w:

$$w \approx \pm |m_1|\, |m_2| \cdots |m_n| \tag{37}$$

(or, more precisely, that the right scaling is obtained; we shall come back to this later). Large deviations theory (expressed here somewhat loosely) tells us that, for large n and given $h < 0$,

$$\mathrm{Prob}\{a_1 + a_2 + \cdots + a_n \approx nh\} \sim 2^{-nF(h)}, \tag{38}$$

where $F(h)$ is the Cramér (or rate) function, defined here to be positive to ensure consistency with the definition of the codimension used in Section 2 (this is also why we use powers of 2 and the minus sign in the definition of a_i).

From (37) and (38), and after integration over all possible h's, we obtain

$$\mathrm{Prob}\{|w| > \xi\} \propto \xi^{-p_\star}, \tag{39}$$

$$p_\star = \inf_{h<0} \frac{F(h)}{-h}. \tag{40}$$

Similarly, for $p \geqslant 0$, we find

$$\langle (|m_1|\, |m_2| \cdots |m_n|)^p \rangle = 2^{-n\zeta_p}, \tag{41}$$

$$\zeta_p = -\ln_2 \langle |m|^p \rangle = \inf_h \,(ph + F(h)). \tag{42}$$

Note that (42) is just Cramér's result that $F(h)$ is the Legendre transform of the logarithm of the characteristic function of the variables a_i. Note also that (39), (40), (41) and (42) resemble (7), (8), (9) and (10) established in Section 2. It is easily shown by manipulations resembling those of Section 2 that $p_\star$ has exactly the value predicted by (35) and that the minimum in (40) is achieved at

$$h_\star = -\langle |m|^{p_\star} \ln_2 |m| \rangle. \tag{43}$$

Obviously, in order for $p_\star$ to exist, negative values for h must be permitted or, in other words, the variable $|m|$ must be allowed to take values in excess of unity.

In evaluating the tail of the series (32) we have so far assumed that it suffices to take a single term of suitable order n. Two questions arise: (i) how large should n be and (ii) what happens if we take the whole series? The answer to the former follows from (36), (37) and (38): since $|w| \approx 2^{-nh_\star}$, we need to have n of the order of $\ln_2 |w|/|h_\star|$. We observe that, the larger $|h_\star|$, the faster (in n) large $|w|$-values will be reached. Consistently with the definition given in Section 2 we call $C \equiv |h_\star|$ the *multifractality parameter*. For the latter question we just briefly report in words our findings which are a bit too technical for the present paper: each of the terms in (32) is, except for a random sign factor, an exponential of a sum of a_i terms. These sums perform a random walk. If, for large n, we put the condition that there should be a large deviation such that the slope of the walk is approximately $h \neq \langle a \rangle$, then the walk is itself close to a straight line of slope h (with small Gaussian fluctuations around it). One then finds the same scaling result as given by (39) and it may be checked that the inclusion of more than one term just affects the constant in front of the power law.

3.2. Multifractal Properties of the Random Multiplicative Process

The fact that the rmp's, as defined at the beginning of Section 3, are multifractal has already been proved in ref. 3. The ζ_p function given there (our Eq. (29)) is precisely the same as (42) for the Kesten map. This is, of course, not accidental since the rmp was introduced by analogy with random multiplicative cascade models, for which a simple relation is known to exist between their multifractal properties and the large deviations of the logarithms of the multipliers (see, e.g., Section 8.6 of ref. 6).

Now we turn to tails of the fractional derivatives of rmp's and show that they have precisely the properties conjectured in Section 2. We shall limit ourselves to instances where the mean slope of the associated Kesten map is greater than one or, equivalently using (35), $p_\star < 1$. Let us make a preliminary remark regarding rmp's for which the largest multiplier is exactly unity, as is the case of the example considered in Section 4 of ref. 3, when account is made for a factor $\sqrt{2}$ in Eq. (9) of that reference. In such a case, neither the associated Kesten map nor the rmp itself can have a power-law tail for w: if the multiplier 1 has finite probability, say $\rho < 1$, after n iteration, w is at most $O(n)$ with probability ρ^n. If we now take a fractional derivative of order $\alpha > 0$, we have already noted that this rescales the multipliers by a factor 2^α so that power-law tails are no more ruled out. In the Kesten map associated to $D^\alpha w$ there will be a power-law tail with exponent $-p_\star$ given, in view of (35), by

$$\langle 2^{p_\star \alpha} |m|^{p_\star} \rangle = 1. \tag{44}$$

By (42) this is equivalent to $\zeta_{p_\star} = p_\star \alpha$, which is identical to (13). For the full rmp we do not know a rigorous derivation of this relation but we shall now show that if $D^\alpha w(x)$ has, for all x, a power-law tail probability with exponent $-p_\star$ $(0 < p_\star < 1)$, then $p_\star$ is given by (44) and the same exponent applies to the global homogeneous process $D^\alpha u$.

For this, following rather closely an argument given in ref. 20, consider

$$K_\alpha(z, x) \equiv \langle e^{izD^\alpha w(x)} \rangle, \tag{45}$$

the characteristic function of the fractional derivative of the local rmp. By (27) and the independence of $m^{(1)}$, $m^{(2)}$, $w^{(1)}$ and $w^{(2)}$, the following integral equation is obtained

$$K_\alpha(z, x) = \langle e^{izD^\alpha \phi(x)} \rangle \int_{\mathbb{R}} K_\alpha(2^\alpha y, 2x)\, p(y)\, dy \int_{\mathbb{R}} K_\alpha(2^\alpha y, 2x-1)\, p(y)\, dy, \tag{46}$$

where $p(\cdot)$ is the probability density of m. By the assumption made, there is a small-z expansion of the form:

$$K_\alpha(z, x) = 1 + |z|^{p_\star}\, g_\alpha(x) + \text{h.o.t.}, \tag{47}$$

where h.o.t. (higher order terms) stands for $o(|z|^{p_\star})$. Expanding (46), collecting all $O(|z|^{p_\star})$ terms and integrating overs x, we find that

$$g_\alpha = g_\alpha \int_{\mathbb{R}} 2^{p_\star \alpha}\, |y|^{p_\star}\, p(y)\, dy, \tag{48}$$

where $g_\alpha \equiv \int_{\mathbb{R}} g_\alpha(x)\, dx$. Eq. (44) follows immediately from (48). From a small-z expansion of (26), with u and w replaced by $D^\alpha u$ and $D^\alpha w$, respectively, it follows that $D^\alpha u$ has the same tail exponent $-p_\star$.

Finally, we briefly indicate the changes in tail probabilities which occur when the order of fractional differentiation α varies. For simplicity we assume that m is bounded and, without loss of generality, that the largest value $m_{\max} = 1$ (otherwise change α into $\alpha - \ln_2 m_{\max}$). For $\alpha = 0$ we already noted that no power-law tail can be present. The down convexity of the function $p_\star \mapsto 2^{\alpha p_\star} \langle |m|^{p_\star} \rangle$ implies that (44) has at most one solution other than $p_\star = 0$. Whether or not is has one depends on the Lyapunov exponent of the Kesten map with multiplier $2^\alpha m$. If $0 < \alpha < -\langle \ln_2 |m| \rangle$ there is a power-law tail. If $\alpha > -\langle \ln_2 |m| \rangle$ there is runaway.

3.3. Designing Numerical Experiments with the Random Multiplicative Process

In this section we perform numerical experiments with a class of rmp's, called dyadic, where the multipliers take only two values (and their opposites). The example considered in Section 4 of ref. 3, here called the *Roman rmp*, is an instance. It has

$$m = \begin{cases} \pm 2^{-1/3}, & \text{each with probability } 7/16; \\ \pm 1, & \text{each with probability } 1/16. \end{cases} \tag{49}$$

The general class studied will be

$$m = \begin{cases} \pm m_1, & \text{each with probability } q/2; \\ \pm 1, & \text{each with probability } (1-q)/2, \end{cases} \tag{50}$$

where $0 < m_1 < 1$ and $0 < q < 1$. It is easily shown that for the Kesten map associated to u the ζ_p's and the multifractality parameter are given by

$$\zeta_p = -\ln_2(qm_1^p + 1 - q), \tag{51}$$

$$C(\alpha) = -\frac{\ln_2(qm_1^{p_\star} + 1 - q)}{p_\star} + \frac{qm_1^{p_\star} \ln_2 m_1}{qm_1^{p_\star} + 1 - q}, \tag{52}$$

where α and $p_\star$ are related by (44). From (51) we can calculate the minimum and maximum Hölder exponents:

$$h_{\min} = 0, \qquad h_{\max} = -q \ln_2 m_1. \tag{53}$$

The simulations reported hereafter have $n \leqslant 16$ iterations of the rmp (but much more for the Kesten maps). The width of the Mexican hat is always $\sigma = 0.1$. For the construction of the local rmp, the simulation interval, from -0.7 to 1.3, is chosen such that the local rmp $w(x)$ is down to (single-precision) roundoff level at the edges of this interval. The resolution is $2^{-(n+4)}$ (the $+4$ is needed because the width of the Mexican hat is substantially smaller than unity). The global rmp is constructed by adding typically one hundred independent Poisson-shifted local realizations with a density of the Poisson distribution $c = 0.5$. Fractional derivatives are calculated as multiplications by $|k|^\alpha$ of the (discrete) Fourier transform (windowing not needed because of the rapid falloff at the edges). Cumulative probabilities are calculated from the global rmp using rank ordering to avoid binning.[21, 22] Cumulative probabilities for Kesten maps are obtained by Monte-Carlo simulation of the random map with a number of iterations ranging from $n = 13$ to $n = 10^4$ and typically 10^4 realizations. Figure 2 shows a graph of one realization of the (local) Roman rmp for $n = 13$. This has $h_{\max} = 7/24 \approx 0.29$. Figure 3(a) shows the cumulative probability of the fractional derivative for an α chosen to give $p_\star = 1/2$ and $n = 16$. No power-law tail is observed. The associated Kesten map does reveal the

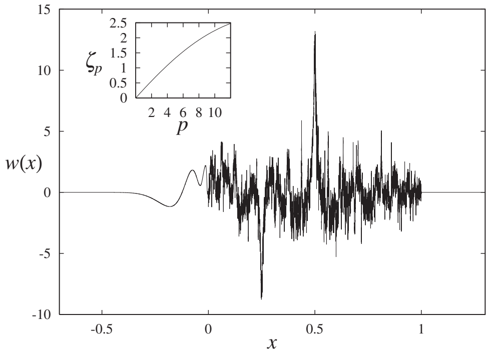

Fig. 2. The (local) Roman rmp[3] and its ζ_p function (as inset).

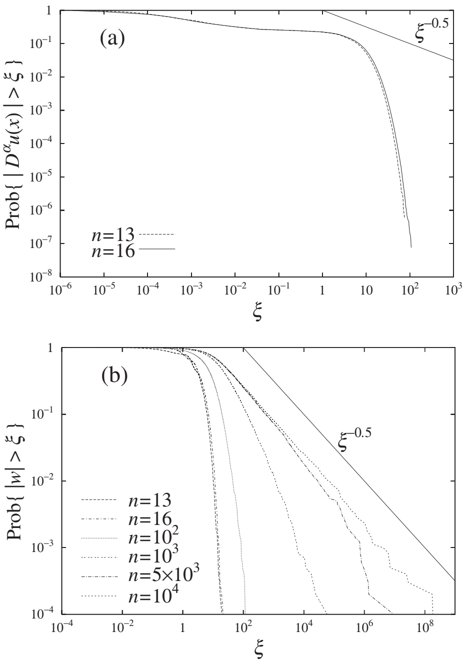

Fig. 3. (a) Distribution of fractional derivative of order $\alpha = 0.2895$ ($p_\star = 0.5$) for the Roman rmp.[3] Number of iterations (levels) n in the Monte-Carlo simulation with 10^2 realizations as labeled. The slope of the line is the predicted asymptotic behavior. No power-law is observed. (b) Distribution for the associated Kesten map (31) with 10^4 realizations. Power-law behavior is observed only when the number of iterations $n \geqslant 5 \times 10^3$.

power-law tail but only after about 5000 iterations (in the rmp this would correspond to the smallest scale being 2^{-5000}!).

The reason for this slow convergence is the exceedingly small value of the multifractality parameter $C = 0.0022991$. Hence, the Roman rmp is not a good candidate for seeing power-law tails in fractional derivatives.

For obtaining better candidates we systematically search through parameter space, varying m_1, q and α and trying to maximize the multifractality parameter $C(\alpha)$ given by (52). In practice, after delineating the acceptable ranges of these parameters, we randomly choose 3×10^4 triplets (m_1, q, α) and each time calculate $C(\alpha)$. We thus obtain about half a dozen triplets for which $C(\alpha)$ is close to unity, i.e., more than 400 times larger than the value for the Roman rmp. We then select the one for which the Kesten map displays a conspicuous power-law tail for n as low as possible. This turns out to be $n \approx 13$ (around $n \approx 10$ a power-law tail begins to emerge). We call this the "optimal rmp." Its parameters are:

$$m_1 = 0.04, \qquad q = 0.6, \qquad \alpha = 1.368, \qquad C(\alpha) = 0.9867. \tag{54}$$

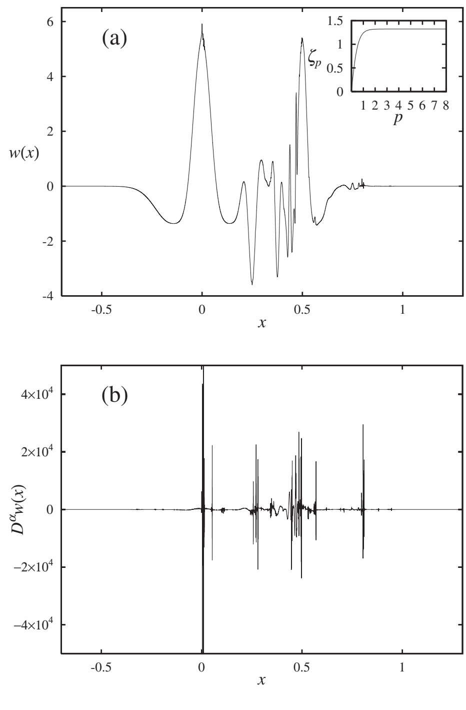

Fig. 4. (a) The (local) optimal rmp defined by (54) and its ζ_p function (as inset); (b) Fractional derivative of order $\alpha = 1.368$ applied to this process; note the spikes.

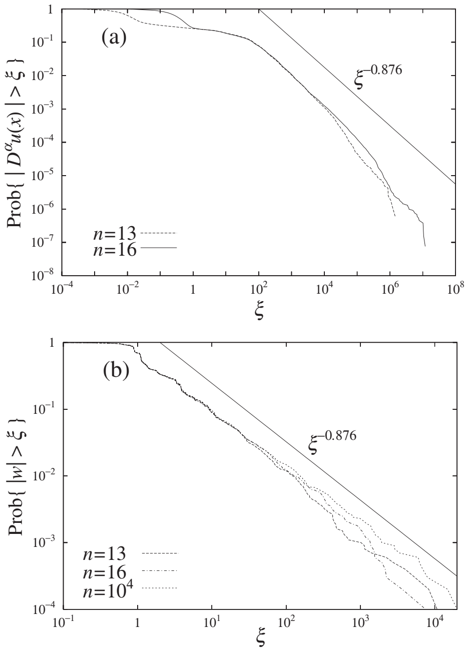

Fig. 5. As in Fig. 3, but for the optimal rmp (54) and $\alpha = 1.368$ ($p_\star = 0.876$). Note the conspicuous power-law tail after about 13 to 16 iterations.

Figure 4 shows the local optimal rmp and its fractional derivative of order $\alpha = 1.368$, which gives $p_\star = 0.876$. Note that the graph of ζ_p (inset on Fig. 4(a)) is much more curved than that for the Roman rmp. Figure 5(a) shows the cumulative probability which is seen to display a power-law tail already for $n = 13$ and $n = 16$. Note that the cumulative probability for the associated Kesten map Fig. 5(b) is a somewhat wiggly power law due to the presence of log-periodic subdominant corrections;[15] this phenomenon is less pronounced if we use lower values of $p_\star$ (e.g., 0.5, not shown).

4. ANALYSIS OF EXPERIMENTAL DATA

As already stated in the Introduction, multifractality was introduced to interpret experimental turbulence data, specifically the up convex bending of the graph of ζ_p. Some of the best high-Reynolds turbulence data have been collected at the ONERA wind tunnel by the Grenoble group from LEGI. The Taylor-scale based Reynolds number is $R_\lambda \approx 2700$ with about three decades of scaling (for definitions see, e.g., ref. 6). We use (longitudinal) velocity data obtained from this group to check for the

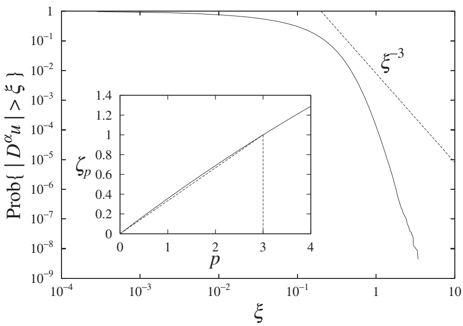

Fig. 6. Distribution of fractional derivative of order $\alpha = 1/3$ for the Modane turbulence data. The power law with exponent $-p_\star = -3$ is the predicted tail behavior at infinite Reynolds number. In fact, no scaling region is observed. Inset: corresponding ζ_p graph which is close to its Kolmogorov 1941 value, a straight line of slope $1/3$.

possible presence of power-law tails in fractional derivatives. A total of 2.3×10^9 data points (in 12 segments of variable lengths) is analyzed. To calculate the fractional derivatives of these non-periodic data we use Hann windowing (see, e.g., Section 13.4 of ref. 23). The different segments are processed separately and the union of all fractional derivative data is then rank ordered to produce cumulative probabilities (using just a single segment does not produce significantly different results; as we shall see the problem we shall encounter has little to do with noise which can be reduced by using more data).

Figure 6 shows the probability of the fractional derivative of order $1/3$, a value chosen because, by Kolmogorov's four-fifths law, the corresponding $p_\star$ should be exactly three.[6] Figure 7 shows the case $\alpha = 13/36$, which corresponds to $p_\star = 1/2$ (when using the lognormal law $\zeta_p = (p/3) + (\mu/18)\, p(3-p)$ with $\mu = 0.2$, an excellent approximation for p up to at

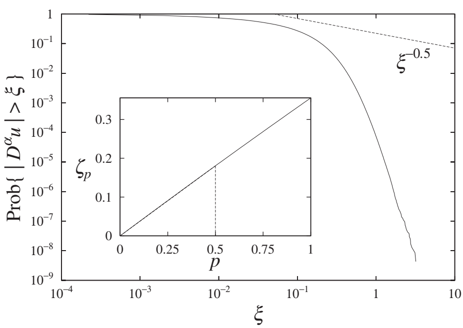

Fig. 7. As in Fig. 6 but for $\alpha = 13/36$ ($p_\star \approx 1/2$).

least six). For both instances we show the nearly straight graph of ζ_p over the relevant range of p's. In neither case do we see any trace of the predicted power law tail.

Here, a digression is in order. The fractional derivative analysis of this turbulence data was actually made by us before the work on random multiplicative processes reported in Section 3. The latter was motivated in part by our desire to interpret the failure to see power laws in experimental data. Of course, now it is clear why we do not see such power laws. The multifractality parameter of the experimental data, using the lognormal approximation, is

$$C(\alpha) = \frac{2+\mu}{6} - \alpha. \tag{55}$$

For $\alpha = 1/3$ we have $C \approx 0.033$ while for $\alpha = 13/36$ we have $C \approx 0.0055$. These values are far too small for power-law tails to be observable with a three-decade long inertial range. Actually, what we observe in Figs. 6 and 7 at high values of the modulus ξ of the fractional derivative is due to the viscous cutoff and can be predicted by the argument of ref. 9, adapted to fractional derivatives.

Higher Reynolds number data with four decades of scaling will soon become available thanks to the use of large-scale cryogenic Helium facilities.[24] This increase is unlikely to be sufficient to see the predicted power laws. Alternative methods, based on the processing of the local dissipation measure by fractional derivatives or integrals should also be explored.

We now consider financial data for which the observation of power-law tails for fractional derivatives could be easier, although the data are far scarcer than in turbulence. Stock market indices are generally believed to display multifractal scaling. We refer, for example, to http://www.ncrg.aston.ac.uk/~vicenter/financial.html which contains a number of references to multifractal (or spuriously multifractal) behavior of financial time series. We use S&P Futures intraday (minute-by-minute) data with linear interpolation for the approximately 12% of missing data points due to lack of activity. The total number of points including interpolated data is 1.88×10^6. We analyze their logarithms (because returns are defined as differences of logarithms). Hann windowing is used again (higher-order windowing makes no substantial difference.). Structure functions (not shown) of order up to $p = 4$ display almost three decades of scaling. The corresponding ζ_p function is shown as an inset on Fig. 8. For $\alpha = 0.348$ we show the cumulative probability (by rank ordering) in Fig. 8. The ζ_p graph is much more curved than that for turbulence data. The multifractality parameter, $C(0.348) \approx 0.30$, being almost one order of magnitude larger

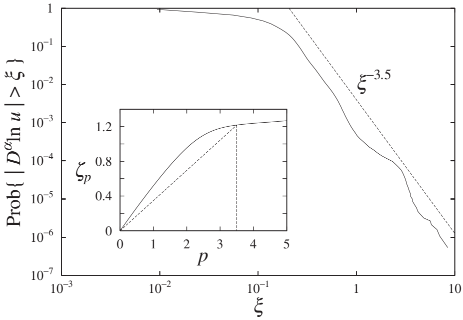

Fig. 8. Distribution of fractional derivative of order $\alpha = 0.348$ ($p_\star = 3.5$) for S&P 500 Futures intraday financial data (minute-by-minute, 1982 April 21—1999 February 26). The bump around $\xi = 2$ is mainly coming from the vicinity of the crash of October 1987. Inset: corresponding ζ_p graph and line of slope α.

than for the turbulence data, it is not surprising to observe a power-law range ($\propto |\xi|^{-3.5}$) in the cumulative probability. Let us finally point out that, in one respect, such financial data are closer to Burgers turbulence (which easily displays power-law tails for fractional derivatives) than to Navier–Stokes turbulence. Indeed, there are occasional strong discontinuities, such as financial crashes.

5. CONCLUSION

We have shown in this paper that multifractality of a function has a signature in the tail behavior of probabilities of fractional derivatives. This signature may frequently be invisible, since power-law tails emerge only if the number of scaling octaves times the multifractality parameter exceeds a value found empirically to be around 10. We recall that the multifractality parameter C is a measure of how strongly the functions departs from being self-similar. Incompressible three-dimensional turbulence having $C \approx 1/30$ is not a good candidate for seeing such tails. There are many other turbulence-like problems displaying multifractality, such as passive or active scalars and magnetohydrodynamic turbulence. Most of these should have larger C's.

It is known that the standard way of detecting multifractality, by structure functions, can produce spurious multifractality. One reason is that self-similarity does already produce structure functions with scaling, albeit with ζ_p linearly proportional to the order p. If there is a mechanism, such as contamination by subdominant terms[25] or very slow convergence,[26, 27] which slightly changes the apparent value of ζ_p, a self-similar function may look multifractal. Such a phenomenon is ruled out with our new method which detects only very robust forms of multifractality.

We mention here some open mathematical issues. Jaffard has recently shown that multifractality is a (topologically) generic phenomenon in many function spaces of the Sobolev family such as Besov spaces, irrespective of any underlying dynamics.[29, 30] Can such a proof be extended to show genericity of power-law tails? For random multiplicative processes it should be possible to give a rigorous proof of some of the results presented somewhat heuristically in Section 3. We notice also that the functional random map formulation (23) can be used to investigate further the statistics of individual Fourier modes, studied in ref. 31.

Finally, returning to three-dimensional turbulence we cannot resist asking: since experimentally this turbulence does not "test positive" when looking for power-laws tails in fractional derivatives does it have other features which would reveal themselves only at vastly larger—and therefore perhaps inaccessible—Reynolds numbers? Could it be that the velocity itself (without taking any derivative) or that velocity increments have power-law tails in their probabilities, as suggested for example in ref. 28? This cannot be ruled out completely. For example, if we accept the log-normal model (with $\mu \approx 0.2$) also for large values of p (for which the exponent h is negative), we find that ζ_p vanishes for $p \approx 33$. This implies a power-law tail with exponent -33. The multifractality parameter is $C \approx 0.37$. Such an algebraic tail, if it exists, would require about 30 octaves or 9 decades of scaling, that is R_λ's of about eighteen millions. Getting there requires the crossing of a substantial "turbulence desert," but a small one in comparison with the great high-energy desert predicted by Giorgi's and Glashaw's grand unified theory.[32]

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